de Luis,[1] to write a simple expression of the axial strain at the interface, valid for high values of T,

$$\epsilon\,|_{z=t_b/2} = \frac{\alpha}{\alpha+\psi}\,\Lambda_{a1}$$

In the case of symmetric actuator patches, $\alpha = 2$ for pure extension and $\alpha = 6$ for pure bending; for an actuator on one side only $\alpha = 4$, as shown before. In the present formulation the fully extensional-bending coupling was accounted for; in fact, the axial deformation is always present in the case of one-sided actuator.

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Computation of Eigenvector Derivatives with Repeated Eigenvalues Using a Complete Modal Space

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Nomenclature

$\bar{H}_h, H_h$	= equivalent high-order modal matrix
K	= real symmetric stiffness matrix
$K_{,j}$	= derivative of stiffness matrix with respect to p_j
$K_{,jj}$	= second-order derivative of stiffness matrix
M	= real symmetric mass matrix
$M_{,j}$	= derivative of mass matrix with respect to p_j
$M_{,jj}$	= second-order derivative of mass matrix
$Z_{,j}$	= derivative matrix of eigenvector of repeated root with respect to design parameter p_j
Λ	= complete eigenvalue diagonal matrix
$\bar{\Lambda}$	= repeated eigenvalue diagonal matrix
$\bar{\Lambda}_{,j}$	= derivative diagonal matrix of repeated eigenvalue with respect to p_j
$\bar{\Lambda}_{,jj}$	= diagonal matrix of second-order derivative of repeated eigenvalue
$\bar{\lambda}$	= repeated eigenvalue
Φ	= complete eigenvector matrix
Φ_h, Λ_h	= high-order eigenpair matrix
Φ_ℓ, Λ_ℓ	= lower order eigenpair matrix
$\bar{\Phi}$	= equivalent complete modal matrix
φ	= eigenvector (column vector)
Ψ, Z	= eigenvector matrix of repeated eigenvalue

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I. Introduction

EIGENVECTOR derivatives are important for structural dynamics, optimal analysis, and control system design. Recently, eigenvector derivative data have become necessary for use in structural model modification,[1–4] system parameter identification,[5–6] and failure diagnosis.[7] In addition, the calculation of eigenvector derivatives has an important role in sensitivity analysis. At present, we have several procedures[8–19] for computing eigenvector derivatives. The governing equation of the eigenvector derivative is defined as

$$(K - \lambda_i M)\varphi_{i,j} = (\lambda_i M_{,j} + \lambda_{i,j} M - K_{,j})\varphi_i = g_i \qquad (1)$$

The coefficient matrix $(K - \lambda_i M)$ is degenerate for m repeated eigenvalues and has rank n–m. In Refs. 8 and 9, the authors solved Eq. (1) by adding one or m independent equations to Eq. (1) through the relation of mass orthogonality. In Refs. 10–15 the authors introduced a procedure to divide the solution technique into two steps: first solve the particular solution by exerting one or m given restrictions on Eq. (1), then find the constants existing in the general solution through the relation of mass orthogonality. In Ref. 15 the author pointed out that the procedure for solving the particular solution in Refs. 11–13 may fail under certain circumstances. Employing the more rigorous method in Ref. 14, this failure can be avoided. In Refs. 16 and 17 the particular solution is yielded by using the matrix disturbance method. In comparison with Nelson method[10] and extended Nelson methods,[11–14] the direct perturbed (DP) method described in Refs. 16 and 17 has the advantages of numerical simplicity and efficiency for solving the particular solution and possibly makes the particular solution approach the general solution directly.

The modal method used in Ref. 8 avoided directly solving Eq. (1). The modal method presented in Ref. 8 was based on the incomplete modal space $\varphi_k (k = 1, 2, \ldots, \ell < n)$. The eigenvector derivative is expressed as

$$\varphi_{i,j} = \sum_{k=1}^{\ell} q_k \varphi_k \qquad (2)$$

where ℓ is the number of lower order modes, and n is the total number of degrees of freedom of the analytic model. The precision of the eigenvector derivatives using the incomplete modes φ_k is poor. To reduce the modal truncation error, the author of Ref. 18 proposed an approach to add a static solution to the incomplete modes to improve the solution,

$$\varphi_{i,j} = \varphi_{i,j}^{(0)} + \sum_{k=1}^{\ell} q_k \varphi_k \qquad (3)$$

where $\varphi_{i,j}^{(0)}$ can be obtained from Eq. (1) with λ_i equal to zero as shown in Eq. (4),

$$K\varphi_{i,j}^{(0)} = g_i \qquad (4)$$

when K is singular, we utilize the "dynamic solution" method for $\varphi_{i,j}^{(0)}$ as proposed in Refs. 16 and 17 and shown as

$$[K - (1+\varepsilon)\lambda_i M]\varphi_{i,j}^{(0)} = g_i \qquad (5)$$

in which ε is a small perturbation parameter. The application of the incomplete modal expansion method with static correction described by Eqs. (3) and (4) to the calculation of eigenvector derivatives with repeated eigenvalues is presented in the Appendix. The development of the incomplete modal expansion methods was described in Ref. 19.

This paper presents a complete modal space (CMS) method that is an exact modal expansion method. There is no modal truncation error existing in the derivation. Also, there is no additional effort required to find $\varphi_{i,j}^{(0)}$ as shown in Eq. (3). Moreover, CMS can be used in the case of repeated eigenvalues. As already mentioned, this method gives better results compared to those obtained from Ref. 8 and Eq. (3).

II. Complete Modal Space Method

Given the symmetric real matrices K and M in $R^{n,n}$, the generalized eigenvalue problem and mass-orthogonality condition are

$$K\Phi = M\Phi\Lambda \tag{6}$$

$$\Phi^T M \Phi = I \tag{7}$$

If $\bar{\lambda}$ is an eigenvalue of multiplicity m and the corresponding m eigenvectors are Ψ which is in $R^{n,m}$ then we have

$$\begin{aligned}\Phi &= \left[\Psi, \Phi_r \vdots \Phi_h\right] = \left[\Phi_\ell \vdots \Phi_h\right] \\ \Lambda &= \mathrm{diag}\left[\bar{\Lambda}, \Lambda_r, \vdots \Lambda_h\right] = \mathrm{diag}\left[\Lambda_\ell \vdots \Lambda_h\right]\end{aligned} \tag{8}$$

where $\bar{\Lambda} = \bar{\lambda} I$, and I is the unit matrix. Φ_r in $R^{n,r}$ are r lower order modes except for Ψ and are determined by Eq. (6). Φ_h in $R^{n,h}$ and Λ_h in $R^{h,h}$ are high-order eigenpairs. Note, Φ_ℓ and Λ_ℓ are obtained from Eq. (6) and Φ_h from the following procedure. Obviously, Ψ satisfies

$$K\Psi = M\Psi\bar{\Lambda} \tag{9}$$

$$\Psi^T M \Psi = I \tag{10}$$

Now, define

$$Z = \Psi\Gamma \tag{11}$$

in which Γ in $R^{m,m}$ is directly orthogonal,

$$\Gamma^T\Gamma = I \tag{12}$$

Since the eigenspace Ψ is degenerate, any linear combination $\Psi\Gamma$ of the columns of Ψ are also eigenvectors of $\bar{\lambda}$. Thus, Z satisfies

$$KZ = MZ\bar{\Lambda} \tag{13}$$

$$Z^T M Z = I \tag{14}$$

Differentiating Eq. (13) with respect to design parameter p_j yields

$$(K - \bar{\lambda}M)Z_{,j} = MZ\bar{\Lambda}_{,j} - (K_{,j} - \bar{\lambda}M_{,j})Z = F \tag{15}$$

where the subscript $_{,j}$ indicates the derivative with respect to p_j. Substituting Eq. (11) into Eq. (15) and left multiplying Eq. (15) by Ψ^T, the standard eigenvalue equation is produced as

$$\left[\Psi^T(K_{,j} - \bar{\lambda}M_{,j})\Psi\right]\Gamma = \Gamma\bar{\Lambda}_{,j} \tag{16}$$

Solving Eq. (16) gives the solutions of $\bar{\Lambda}_{,j}$, Γ, and Z. The solution of $\mathbf{Z}_{,j}$ remains to be found.

The key step of CMS is to express $Z_{,j}$ as a combination of complete modes

$$Z_{,j} = [Z, \Phi_r, \Phi_h]\begin{bmatrix} Q \\ Q_r \\ Q_h \end{bmatrix} \tag{17}$$

As is well known, it is difficult to obtain the high-order modes Φ_h by solving Eq. (6). Now let us introduce equivalent modal matrix $\bar{H}_h$ in $R^{n,n}$, including the high-order mode information. From Eq. (7), one can get

$$\begin{aligned}M^{-1} &= \Phi\Phi^T \\ &= \Phi_\ell\Phi_\ell^T + \Phi_h\Phi_h^T\end{aligned} \tag{18}$$

M is a positive matrix. The definition of the matrix $\bar{H}_h$ is $\Phi_h\Phi_h^T M$. Equation (19) can be obtained from Eq. (18)

$$\bar{H}_h \equiv \Phi_h\Phi_h^T M = I - \Phi_\ell\Phi_\ell^T M \tag{19}$$

The rank of matrix $(\Phi_h\Phi_h^T)$ is h and the rank of M is n, therefore, the rank of matrix $\bar{H}_h$ equals h, where $h = n - \ell$. There are only h independent columns (or rows) in the matrix H_h, because the dimension of matrix $\bar{H}_h$ is $n \times n$. Although these h independent columns (or rows) are nonunique in the matrix $\bar{H}_h$, there are always ℓ dependent columns (or rows) in the matrix $\bar{H}_h$. From the mathematical point of view, one can use any numerical analysis technique to eliminate ℓ dependent columns (or rows) from the matrix $\bar{H}_h$, in order to obtain an equivalent high-order modal matrix $H_h \in R^{n,h}$. Both matrices H_h and Φ_h are entirely different in matrix elements, but they physically make the same contribution. The mathematical derivation showing that both matrices H_h and Φ_h equivalently have the same contribution is shown in Ref. 20. It can be proved that

$$\Phi_\ell^T M H_h = 0, \qquad \Phi_\ell^T K H_h = 0 \tag{20}$$

For this, the following conditions are first demonstrated:

$$\Phi_\ell^T M \bar{H}_h = 0, \qquad \Phi_\ell^T K \bar{H}_h = 0 \tag{21}$$

Embedding Eq. (19) into Eq. (21) yields

$$\begin{aligned}\Phi_\ell^T M\left(I - \Phi_\ell\Phi_\ell^T M\right) &= \Phi_\ell^T M - \Phi_\ell^T M = 0 \\ \Phi_\ell^T K\left(I - \Phi_\ell\Phi_\ell^T M\right) &= \Phi_\ell^T K - \Lambda_\ell\Phi_\ell^T M = 0\end{aligned} \tag{22}$$

Because h column vectors in H_h belong to a part of n column vectors of $\bar{H}_h$, Eq. (20) holds true. Substituting H_h into Eq. (17) to replace Φ_h gives

$$\mathbf{Z}_{,j} = [Z, \Phi_r, H_h]\begin{bmatrix} Q \\ Q_r \\ Q_h \end{bmatrix} = \tilde{\Phi}\tilde{Q} \tag{23}$$

Embedding Eq. (23) into Eq. (15), left multiplying it by $\tilde{\Phi}^T$, and resulting from Ψ, Φ_r, and H_h all being orthogonal with respect to K and M, we will have

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_r - \bar{\lambda}I & 0 \\ 0 & 0 & \kappa - \bar{\lambda}\mu \end{bmatrix}\begin{bmatrix} Q \\ Q_r \\ Q_h \end{bmatrix} = \tilde{\Phi}^T F = \begin{bmatrix} 0 \\ \tilde{F}_r \\ \tilde{F}_h \end{bmatrix} \tag{24}$$

in which

$$\kappa = H_h^T K H_h, \qquad \mu = H_h^T M H_h \tag{25a}$$

$$\tilde{F}_r = -\Phi_r^T(K_{,j} - \bar{\lambda}M_{,j})Z \tag{25b}$$

$$\tilde{F}_h = -H_h^T(K_{,j} - \bar{\lambda}M_{,j})Z \tag{25c}$$

From Eq. (24) we have

$$Q_r = \left(\Lambda_r^{-1} - \bar{\lambda}^{-1}I\right)\tilde{F}_r \tag{26}$$

$$(\kappa - \bar{\lambda}\mu)Q_h = \tilde{F}_h \tag{27}$$

In order to obtain the factor Q, differentiating Eq. (14) with respect to p_j yields

$$\mathbf{Z}_{,j}^T MZ + Z^T M_{,j} Z + Z^T M\mathbf{Z}_{,j} = 0 \tag{28}$$

Then substituting Eq. (23) into Eq. (28) leads to

$$Q^T + Q = -Z^T M_{,j} Z = B \tag{29}$$

To determine all nondiagonal elements in the Q matrix, differentiating Eq. (13) twice and left multiplying by Z^T gives

$$\begin{aligned}&Z^T(K_{,jj} - \bar{\lambda}M_{,jj})Z + 2Z^T(K_{,j} - \bar{\lambda}M_{,j})\mathbf{Z}_{,j} \\ &- 2Z^T M_{,j} Z\bar{\Lambda}_{,j} - 2Z^T M Z_{,j}\bar{\Lambda}_{,j} - \bar{\Lambda}_{,jj} = 0\end{aligned} \tag{30}$$

Embed Eq. (23) into Eq. (30) and utilize the relations $Z^T(K_{,j} - \bar{\lambda}M_{,j})Z = Z^T[MZ\bar{\Lambda}_{,j} - (K - \bar{\lambda}M)Z_{,j}] = \bar{\Lambda}_{,j}$ to yield

$$\begin{aligned}&Q\bar{\Lambda}_{,j} - \bar{\Lambda}_{,j}Q + 0.5\bar{\Lambda}_{,jj} \\ &= Z^T(K_{,j} - \bar{\lambda}M_{,j})\Phi_r Q_r + Z^T(K_{,j} - \bar{\lambda}M_{,j})H_h Q_h \\ &- Z^T M_{,j} Z\bar{\Lambda}_{,j} + 0.5Z^T(K_{,jj} - \bar{\lambda}M_{,jj})Z \equiv R\end{aligned} \tag{31}$$

Table 1 Eigenvector derivatives in the case of distinct repeated eigenvalue derivatives

Results using Refs. 9 and 13		Results using CMS method, without Φ_r		Results using incomplete modal method without static correction, with Φ_r		Results using incomplete modal method with static correction, with Φ_r		Results using incomplete modal method with static correction, without Φ_r	
$\bar{\lambda}_{1,j}=0$	$\bar{\lambda}_{2,j}=0.091856$	$\bar{\lambda}_{1,j}=0$	$\bar{\lambda}_{2,j}=0.091856$	$\bar{\lambda}_{1,j}=0$	$\bar{\lambda}_{2,j}=0.091856$	$\bar{\lambda}_{1,j}=0$	$\bar{\lambda}_{2,j}=0.091856$	$\bar{\lambda}_{1,j}=0$	$\bar{\lambda}_{2,j}=0.091856$
0	−0.0020519	0	−0.0020519	0	−0.12060	0	−0.0020514	0	−0.0020344
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0.0051086	0	0.0051086	0	0.29032	0	0.0051054	0	0.0050661
0	0.0012002	0	0.0012002	0	0.087053	0	0.0012003	0	0.0011849
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	−0.0019990	0	−0.0019990	0	0.026197	0	−0.0019988	0	−0.0020294

Table 2 Eigenvector derivatives in the case of identical repeated eigenvalue derivatives

Results using Refs. 9		Results using CMS method, without Φ_r		Results using incomplete modal method without static correction, with Φ_r	
$\bar{\lambda}_{1,j}=0.091856$	$\bar{\lambda}_{2,j}=0.091856$	$\bar{\lambda}_{1,j}=0.091856$	$\bar{\lambda}_{2,j}=0.091856$	$\bar{\lambda}_{1,j}=0.091856$	$\bar{\lambda}_{2,j}=0.091856$
−0.0020519	0	−0.0020519	0	0.14228	0
0	−0.0020519	0	−0.0020519	0.006708	−0.12060
0	−0.0051086	0	−0.0051086	0.016148	−0.29032
0.0051086	0	0.0051086	0	−0.34250	0
0.0012002	0	0.0012002	0	−0.10270	0
0	0.0012002	0	0.0012002	−0.004842	0.08705
0	0.0019990	0	0.0019990	0.001457	−0.02620
−0.0019990	0	−0.0019990	0	−0.030906	0

Let q_{ip}, b_{ip}, and r_{ip} be the elements of the Q, B, and R matrices, respectively. Thus, from Eqs. (29) and (31) and from Refs. 13–15 and 19, it is known that

$$q_{ip}=\begin{cases} r_{ip}/(\bar{\lambda}_{p,j}-\bar{\lambda}_{i,j}), & \bar{\lambda}_{i,j}\neq\bar{\lambda}_{p,j} \\ 0.5b_{ip}, & \text{otherwise}\end{cases} \tag{32}$$

It is pointed out that when $\bar{\lambda}_{p,j}=\bar{\lambda}_{i,j}$ for $p\neq i$, i.e., when there are repeated eigenvalue derivatives, the eigenvector $\boldsymbol{\Gamma}$ in the standard eigenequation Eq. (16) is not unique such that Z and $Z_{,j}$ is not unique. However as long as $q_{ij}+q_{ji}=b_{ij}=b_{ji}$, the resulting $Z_{,j}$ will be a valid derivative as shown in Refs. 13–15.

It should be pointed out that the procedure described by both Eq. (26) and (32) can be considered as the incomplete modal method without static correction for the calculation of eigenvector derivatives with repeated eigenvalues.

III. Discussion

A comparison between the methods described in this paper and that presented in Refs. 11–14 is presented. Although these two methods are completely different, there exists a certain relationship between them. Assume that Φ_ℓ in Eq. (19) includes only the matrix Ψ. This means that the column vectors of matrices Z (i.e., Ψ) and H_h construct a complete modal space. Under this circumstance, H_h includes the contribution of all lower and high-order modes except Ψ. This is the CMS method without Φ_r in the example section. Thus, Eq. (23) can be changed to

$$Z_{,j}=H_hQ_h+ZQ \tag{33}$$

In Refs. 11–14, the general solution is

$$Z_{,j}=V_{,j}+ZC \tag{34}$$

Note, the second terms on the right-hand side of Eqs. (33) and (34) are the same. Also, the first term on the right-hand side of Eq. (33) produces the same contribution as the particular solution $V_{,j}$ as shown in Eq. (34).

It is emphasized that Φ_ℓ can include only Ψ with the form of CMS. Thus, like other methods shown in Refs. 9–17, CMS needs only the eigenvector Ψ and its repeated eigenvalue. In this case, Eq. (26) and the first term of the right-hand side of Eq. (31) can be eliminated. Using Eq. (6) to find the solution of Ψ, Φ_r can be treated as a by-product in the solution of Ψ. If one can insert the computed Φ_r solution into the matrix Φ_ℓ, then the dimension of Eq. (27) can be reduced with Φ_r known, and the computational efforts to find solution of Eq. (27) can be reduced corresponding to the dimension of Φ_r. The more columns of Φ_r we have, the less computing time is needed to solve Eq. (27).

If one needs to compare the CMS method to Ref. 18, which used the static solution to the incomplete modes to improve the solution, then Ref. 18 needs to solve Eq. (4) with incomplete modes, and the CMS method needs to solve Eq. (27), which has a dimension less than Eq. (4). Also, the solution of Ref. 18 is always an approximate solution and can merely guarantee the accuracy of derivatives of the lower order modes. The CMS method is an accurate solution method and can guarantee the accuracy of derivative of any modes.

For a nonrepeated eigenvalue condition, Eqs. (26) and (27) are still valid for the CMS method. But Q_r, Q_h, $\bar{F}_r$, and $\bar{F}_h$ are reduced to column vectors. Q becomes an element q_i and can easily be found via the formula in Ref. 10.

Finally, the present authors point out that if one can find a new matrix H_h which has an orthogonality relation with respect to matrices M and K, then κ and μ on the left-hand side of Eq. (27) will become diagonal. Thus, the effort to find Q_h from Eq. (27) will be as simple as finding Q_r from Eq. (26). To find a new H_h matrix to be orthogonal to M and K, an optimal Gram–Schmidt method was proposed in Ref. 20.

IV. Example

For convenience, the numerical example shown in Ref. 13 is taken as the example in this paper. It forms a finite element model for a square-section cantilever beam, using two 8-degrees-of-freedom beam elements. Such a vibration system has four double repeated eigenvalues, in which the first repeated eigenvalue is equal to 1.8414. The eigenvector derivatives of the first repeated eigenvalue, with respect to section area moment I_z (i.e., the case in Table 1) and $I = I_z = I_y$ (i.e., the case in Table 2), are computed here. To demonstrate the efficiency of the method, we have made the computations using incomplete modal method as shown in Eqs. (26) and (32) as well as the improved incomplete modal method given in the Appendix. The computational cases and the final results are as follows.

Complete Modal Space Method

1) The derivatives of the repeated eigenvalues are distinct both using no extra computed modes Φ_r (Table 1) and using extra computed modes Φ_r. The results are the same as shown in Table 1 without Φ_r. Take the two eigenvectors of the second repeated eigenvalue (73.481) as Φ_r, i.e.,

$$\Phi_r = \begin{bmatrix} -0.069686 & 0 \\ 0 & -0.069686 \\ 0 & -0.16775 \\ 0.16775 & 0 \\ 0.050300 & 0 \\ 0 & 0.050300 \\ 0 & -0.015137 \\ 0.015137 & 0 \end{bmatrix}$$

2) The derivatives of the repeated eigenvalues are identical both using no extra computed modes Φ_r (Table 2) and using extra computed modes Φ_r. The results are the same as shown in Table 2 without Φ_r.

Incomplete Modal Method Without Static Correction

The results are listed in Tables 1 and 2. The incomplete modal space method mentioned here is identical to the Fox method shown in Ref. 8, that is, the CMS method without H_h. The incomplete modal space results shown in Tables 1 and 2 are obtained by using Z, which is constructed by two eigenvectors of the first repeated eigenvalues $\bar{\lambda}_1$ and Φ_r constructed by two eigenvectors of the second repeated eigenvalues $\bar{\lambda}_2$.

Incomplete Modal Method with Static Correction

The results are obtained via the formulas in the Appendix and are listed in Table 1, in which there are two computation cases: 1) No extra computed modes Φ_r exist in the calculation. 2) Extra computed modes Φ_r exist in the calculation. The static residual modes $\Phi^{(0)}$ given by Eq. (A1) are

$$\Phi^{(0)} = \begin{bmatrix} 0 & -0.0020086 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.0050020 \\ 0 & 0.0011699 \\ 0 & 0 \\ 0 & 0 \\ 0 & -0.0020037 \end{bmatrix}$$

As stated in the Appendix, a zero vector in $\Phi^{(0)}$ should be deleted.

The original data (such as M, K, and $K_{,j}$) and the computed intermediate results (such as Γ and Z) can be found in Ref. 13. From Tables 1 and 2, it is concluded that CMS can give the correct eigenvector derivatives of repeated eigenvalues. It does not need to compute any extra modes. In contrast with CMS, in the incomplete modal method without static correction[8] gives unacceptable results even if some extra modes Φ_r are utilized. In addition, the precision of the incomplete modal method with static correction is also less than that of CMS, even if several extra modes Φ_r are employed in the analysis. The reason for these results is that all of the incomplete modal expansion methods are approximate.

V. Conclusion

In this paper an accurate modal expansion method for eigenvector derivatives has been developed. Both results for repeated eigenvalues with distinct and identical eigenvalue derivatives are presented. This method can be very useful for the calculation of eigenvector derivatives with complete modal space.

Appendix: Incomplete Modal Method with Static Correction for Eigenvector Derivatives with Repeated Roots

The principal idea of the incomplete modal method with static correction is to add the static modes $\Phi^{(0)}$ to the incomplete modal expansion. The $\Phi^{(0)}$ are computed via

$$K\Phi^{(0)} = F \tag{A1}$$

Then the eigenvector derivatives $Z_{,j}$ are assumed to be spanned by the eigenvectors Z of concern and the extra mode shapes Φ_r together with the static modes $\Phi^{(0)}$ as follows:

$$Z_{,j} = \left[Z, \Phi_r, \Phi^{(0)}\right] \begin{bmatrix} Q \\ Q_r \\ Q_0 \end{bmatrix} \tag{A2}$$

Note, if $\Phi^{(0)} \in R^{n,m}$ obtained by solving Eq. (A1) contains zero column vectors, then these vectors should be eliminated. Not only is the contribution of these zero vectors null, but also they must lead to the discontinuance of the operation of the computer.

The orthogonality between $\Phi^{(0)}$ and Z with respect to M and K is demonstrated here. For this, $\Phi^{(0)}$ is expressed as

$$\Phi^{(0)} = K^{-1}F = K^{-1}MZ\bar{\Lambda}_{,j} - K^{-1}(K_{,j} - \bar{\lambda}M_{,j})Z \tag{A3}$$

Thus,

$$\begin{aligned} Z^TM\Phi^{(0)} &= Z^TMK^{-1}MZ\bar{\Lambda}_{,j} - Z^TMK^{-1}(K_{,j} - \bar{\lambda}M_{,j})Z \\ &= \bar{\lambda}^{-1}Z^TKK^{-1}MZ\bar{\Lambda}_{,j} - \bar{\lambda}^{-1}Z^TKK^{-1}(K_{,j} - \bar{\lambda}M_{,j})Z \\ &= \bar{\lambda}^{-1}[\bar{\Lambda}_{,j} - Z^T(K_{,j} - \bar{\lambda}M_{,j})Z] \\ &= 0 \end{aligned} \tag{A4}$$

$$\begin{aligned} Z^TK\Phi^{(0)} &= Z^TKK^{-1}MZ\bar{\Lambda}_{,j} - Z^TKK^{-1}(K_{,j} - \bar{\lambda}M_{,j})Z \\ &= \bar{\Lambda}_{,j} - Z^T(K_{,j} - \bar{\lambda}M_{,j})Z \\ &= 0 \end{aligned} \tag{A5}$$

Using the same deductive process, we can prove that there is no orthogonality between $\Phi^{(0)}$ and Φ_r with respect to M and K. Also, the column vectors in $\Phi^{(0)}$ are not orthogonal to each other. That is,

$$\begin{gathered} \Phi_r^TK\Phi^{(0)} \neq 0, \qquad \Phi_r^TM\Phi^{(0)} \neq 0 \\ \Phi^{(0)T}K\Phi^{(0)} = \kappa_0 \neq \text{diagonal matrix} \\ \Phi^{(0)T}M\Phi^{(0)} = \mu_0 \neq \text{diagonal matrix} \end{gathered} \tag{A6}$$

Substituting Eq. (A2) into Eq. (15) yields

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & (\Lambda_r - \bar{\lambda}I) & A_{12} \\ 0 & A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} Q \\ Q_r \\ Q_0 \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{F}_r \\ \tilde{F}_0 \end{bmatrix} \tag{A7}$$

in which

$$\begin{aligned} A_{12} &= \Phi_r^T(K - \bar{\lambda}M)\Phi^{(0)} \\ A_{22} &= \kappa_0 - \bar{\lambda}\mu_0 \end{aligned} \tag{A8}$$

$$\begin{aligned} \tilde{F}_r &= -\Phi_r^T(K_{,j} - \bar{\lambda}M_{,j})Z \\ \tilde{F}_0 &= -\Phi^{(0)T}(K_{,j} - \bar{\lambda}M_{,j})Z \end{aligned} \tag{A9}$$

From Eq. (A7), the following set of equation is given

$$\begin{aligned} (\Lambda_r - \bar{\lambda}I)Q_r + A_{12}Q_0 &= \tilde{F}_r \\ A_{12}^TQ_r + A_{22}Q_0 &= \tilde{F}_0 \end{aligned} \tag{A10}$$

Simultaneously solving Eq. (A10) yields

$$Q_r = \left[I - (\Lambda_r - \bar{\lambda}I)^{-1}A_{12}B^{-1}A_{12}^T\right](\Lambda_r - \bar{\lambda}I)^{-1} \times \tilde{F}_r - (\Lambda_r - \bar{\lambda}I)^{-1}A_{12}B^{-1}\tilde{F}_0 \tag{A11}$$

$$Q_0 = B^{-1}\tilde{F}_0 - B^{-1}A_{12}^T(\Lambda_r - \bar{\lambda}I)^{-1}\tilde{F}_r \tag{A12}$$

where

$$B = A_{22} - A_{12}^T(\Lambda_r - \bar{\lambda}I)^{-1}A_{12} \tag{A13}$$

Because matrix B is of order $m \times m$, the calculation of B^{-1} is easy. In addition, when the number of extra modes Φ_r is not large, adopting the following matrix equation for solving $[Q_r^T, Q_0^T]^T$ is simpler:

$$\begin{bmatrix} \Lambda_r - \bar{\lambda}I & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} Q_r \\ Q_0 \end{bmatrix} = \begin{bmatrix} \tilde{F}_r \\ \tilde{F}_0 \end{bmatrix} \tag{A14}$$

The steps to solve factor Q are the same as that stated in the text, that is, Eqs. (29–32) are used in the calculation of Q provided Eq. (31) is rewritten as

$$R \equiv Z^T(K_{,j} - \bar{\lambda}M_{,j})\Phi_r Q_r + Z^T(K_{,j} - \bar{\lambda}M_{,j})\Phi^{(0)}Q_0 - Z^TM_{,j}Z\bar{\Lambda}_{,j} + 0.5Z^T(K_{,jj} - \bar{\lambda}M_{,jj})Z \tag{A15}$$

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